Real Analysis 2010-09.

Review

- Outer measure ($\mu(\emptyset)=0$, $\mu(A) \le \sum_{i=1}^{\infty} \mu(A_i)$ if $A \subseteq \bigcup_{i=1}^{\infty} A_i$)

- (Caratheodory's Thm) Let μ be an outer measure on X.

Then $(X, \mathcal{M}_c, \mu)$ is a complete measure space,
where $\mathcal{M}_c = \{ E \subset X : E \text{ is } \mu\text{-measurable} \}$.

§ 2.2 Topological and metric spaces.

- Topological spaces
- Metric spaces.

Prop. 2.3 Let $(X, \mathcal{M}, \mu)$ be a measure space, where X is a topological space and assume that

$$\mathcal{M} \supseteq \mathcal{B}_X \quad (\text{where } \mathcal{B}_X \text{ is the Borel } \sigma\text{-algebra on } X)$$

Then any continuous function $f: X \to \mathbb{R}$ (or $\overline{\mathbb{R}}$) is $\mathcal{M}$-measurable.

Pf. $\forall$ open $G \subset \mathbb{R}$ (or $\overline{\mathbb{R}}$), by continuity,

$f^{-1}(G)$ is open in X

hence $f^{-1}(G) \in \mathcal{B}_X \subset \mathcal{M}$. $\square$.

Def. (Borel measure)

An outer measure μ on a topological space X is said to be a Borel measure if all Borel sets are μ-measurable.

Prop 2.4 (Caratheodory's criterion)

Let (X, d) be a metric space. Let μ be an outer measure on X. Suppose that μ satisfies

$$(*) \quad \mu(A \cup B) = \mu(A) + \mu(B) \text{ if } d(A, B) > 0,$$

where $d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$.

(An outer measure μ satisfying $(*)$ is called a metric outer measure)

Then μ is a Borel measure.

Proof. It suffices to show that all closed sets in X are μ-measurable.

Let A be a closed set. We need to show

$$\mu(C) \geq \mu(C \cap A) + \mu(C \setminus A), \quad \forall\, C \subset X.$$

For $n \in \mathbb{N}$, define

$$A_n = \{ x \in X : \ d(x, A) \leq \tfrac{1}{n} \},$$

where $d(x, A) := \inf \{ d(x, y) : y \in A \}$.

Then A_n are closed, $A_n \searrow A$.

Notice that $d(A, A_n^c) \geq \frac{1}{n}$. So

$$d(C \cap A, \ C \setminus A_n) \geq \tfrac{1}{n}.$$

Hence $\mu(C) \geq \mu\big((C \cap A) \cup (C \setminus A_n) \big)$

$$= \mu(C \cap A) + \mu(C \setminus A_n), \quad \forall\, n.$$

Next we show that

$$(**) \qquad \varliminf_{n\to\infty} \mu(C\setminus A_n) \geq \mu(C\setminus A),$$

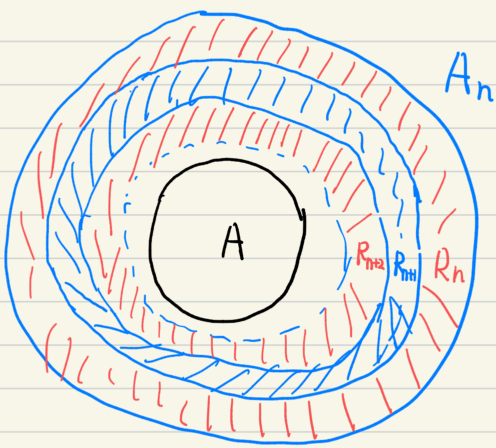
which implies $\quad \mu(C) \geq \mu(C\cap A) + \mu(C\setminus A)$.

To prove $(**)$, define for $k \in \mathbb{N}$,

$$R_k = \left\{ x\in X : \frac{1}{1+k} < d(x,A) \leq \frac{1}{k} \right\}.$$

Then

$$A_n = A \cup \left(\bigcup_{k=n}^{\infty} R_n \right) \quad \text{with union being disjoint.}$$



Then $\quad A^c = A_n^c \cup (A_n \setminus A)$

So

$$C\setminus A \subset (C\setminus A_n)\cup(C\cap(A_n\setminus A))$$
$$=(C\setminus A_n)\cup\left(C\cap\left(\bigcup_{k=n}^{\infty}R_k\right)\right).$$

Hence

$$\mu(C\setminus A)\le\mu(C\setminus A_n)+\mu\left(C\cap\left(\bigcup_{k=n}^{\infty}R_k\right)\right).$$

To show that $\varliminf\limits_{n\to\infty}\mu(C\setminus A_n)\ge\mu(C\setminus A)$,

it is enough to show

$$\sum_{k=1}^{\infty}\mu(C\cap R_k)<\infty$$

(which implies $\mu(C\cap\bigcup_{k=n}^{\infty}R_k)\to 0$ as $n\to\infty$.)

Notice that $R_2, R_4, \cdots, R_{2k}, \cdots$, have positive distance between them, So are $R_1, R_3, R_5, \dots$

Hence

$$
\begin{aligned}
\mu(C) &\geq \mu\big((C\cap R_2) \cup (C\cap R_4) \cup \cdots (C\cap R_{2k})\big) \\
&= \mu(C\cap R_2) + \mu\big((C\cap R_4) \cup \cdots \cup (C\cap R_{2k})\big) \\
&= \cdots \\
&= \mu(C\cap R_2) + \cdots + \mu(C\cap R_{2k}).
\end{aligned}
$$

Hence

$$\sum_{k=1}^{\infty} \mu(C\cap R_{2k}) \leq \mu(C) < \infty.$$

Similarly

$$\sum_{k=1}^{\infty} \mu(C\cap R_{2k-1}) \leq \mu(C) < \infty$$

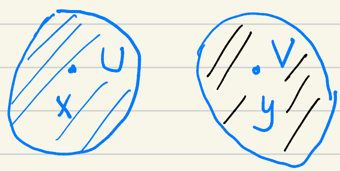
Therefore $\displaystyle\sum_{k=1}^{\infty} \mu(C\cap R_k) \leq 2\cdot\mu(C) < \infty$ $\square$

§ 2.3 Locally compact Hausdorff spaces.

Def. A topological space X is said to be a Hausdorff space if $\forall\ x, y \in X$ with $x \neq y$,

$\exists$ open sets U and V such that

$$x \in U, \quad y \in V \quad \text{and} \quad U \cap V = \emptyset.$$



Def. A topological space is said to be locally compact if $\forall\ x \in X$, $\exists$ an open U such that

$x \in U$ and $\overline{U}$ is compact.

(closure of U).

Notation: X is said to be a LCHS if X is a Hausdorff space and locally compact.

<u>Remark</u> :

- $\mathbb{R}^n$ is a LCHS.
- All compact metric spaces are LCHS.

Prop 2.5. Let X be a LCHS.

Let $K \subset G$ where K is compact, G is open in X.

Then $\exists$ open V such that

$$K \subset V \subset \overline{V} \subset G.$$

Thm 2.6 (Urysohn's lemma).

Let X be a LCHS.

Let $K \subset G$, K compact, G open.

Then there exists a cts $f : X \to \mathbb{R}$ such that

- $\mathrm{supp}(f) \subset G$
- $0 \leq f \leq 1$ on X
- $f(x) = 1$ for all $x \in K$

where

$$\mathrm{supp}(f) = \overline{\{ x : f(x) \neq 0 \}}.$$

Thm 2.7 (partition of Unity).

Let X be a LCHS.

Suppose $K \subset \bigcup_{k=1}^{N} G_k$, with G_k open

where K is compact.

There exist $\{\varphi_j\}_{j=1}^{N} \subset C(X)$,

such that

$$\varphi_j < G_j \quad \text{and} \quad \sum_{j=1}^{N} \varphi_j = 1 \text{ on } K$$

where $\varphi_j < G_j$ means that

① $\operatorname{supp}(\varphi_j) \subset G_j$; and

② $0 \leq \varphi_j \leq 1$ on X.

§ 2.4 Riesz representation Thm.

- For a topologial space X, let

$$C_c(X) = \{ f \in C(X) : \operatorname{supp}(f) \text{ is compact} \}.$$

clearly, $C_c(X)$ is a vector space.

(that is, $af + bg \in C_c(X)$ if $f, g \in C_c(X)$ and $a, b \in \mathbb{R}$)

- A linear functional on a vector space is simply a linear map from the vector space to $\mathbb{R}$.

- A linear functional Λ on $C_c(X)$ is called positive if

$$\Lambda(f) \geq 0 \quad \text{if} \quad f \geq 0.$$

Example: Let X be a topological space.

Let μ be a Borel measure on X such that $\mu(K) < \infty$ for all compact sets K.

Define

$$\Lambda(f) = \int_X f \, d\mu, \quad \forall f \in C_c(X).$$

Then Λ is a positive linear functional on $C_c(X)$.

Justification: It is easy to show the positivity and linearity of Λ.

Below we show that $\Lambda(f) \in \mathbb{R}$ for $f \in C_c(X)$.

Let $f \in C_c(X)$ and $K = \operatorname{Supp}(f)$.

Then K is compact.

Hence

$$\sup_{x \in X} |f(x)| = \sup_{x \in K} |f(x)| < \infty.$$

It follows that

$$\int_X f \, d\mu = \int_K f \, d\mu.$$

and

$$\left| \int_K f \, d\mu \right| \leq \int_K |f| \, d\mu$$

$$\leq \mu(K) \cdot \sup_{x \in K} |f(x)|$$

$$< \infty.$$

Thm 2.8 (Riesz representation Thm)

Let X be a LCHS. Let Λ be a positive linear functional on $C_c(X)$.

Then $\exists$ a Borel measure μ on X such that μ is finite on every compact set, and

$$\Lambda(f) = \int f \, d\mu, \qquad \forall \ f \in C_c(X).$$

- Before the proof, we construct a measure μ from Λ.

Let $G \subset X$ be non-empty and open. We define

$$\mu_0(G) = \sup\{ \Lambda(f) : f \prec G \}.$$

(Recall $f \prec G$ means that $f \in C_c(X)$, $0 \leq f \leq 1$, $\operatorname{supp}(f) \subset G$.
By Urysohn Lem, $\exists$ $f \in C_c(X)$ such that $f \prec G$)

Next set $\mu_0(\emptyset) = 0$.

Now for any $E \subset X$, define

$$\mu(E) = \inf\{\mu_0(G) : G \text{ is open}, G \supset E\}.$$

Proof of Riesz representation Thm (Thm 2.8):

First observe that

$$\mu_0(G_1) \leq \mu_0(G_2) \text{ for open set } G_1, G_2 \text{ with } G_1 \subset G_2.$$

As a direct consequence, we have

(i) $\mu(G) = \mu_0(G)$ for open $G \subset X$.

(ii) $\mu(E_1) \leq \mu(E_2)$ if $E_1 \subset E_2$.

Next we prove the theorem in 4 steps.

① μ is an outer measure.

② all Borel sets are μ-measurable.

③ $\mu(K) < \infty$ for compact K.

④ $\Lambda(f) = \int f \, d\mu$, $f \in C_c(X)$.

Step 1. μ is an outer measure.

We need to show that

$$\mu(E) \le \sum_{j=1}^{\infty} \mu(E_j) \text{ if } E \subset \bigcup_{j=1}^{\infty} E_j.$$

We may assume $\sum_{j=1}^{\infty} \mu(E_j) < \infty$.

Let $\varepsilon > 0$. Pick open $G_j \supset E_j$ such that

$$\mu(E_j) > \mu_0(G_j) - \frac{\varepsilon}{2^j}, \quad j = 1, 2, \ldots$$

Set $G = \bigcup_{j=1}^{\infty} G_j$. Then G is open.

Now we estimate $\mu_0(G)$. Let $f \prec G$.

Let $K = \mathrm{supp}(f)$. Then K is compact.

Since $K \subset G = \bigcup_{j=1}^{\infty} G_j$, by (the) compactness of K,

$\exists\ N$ such that

$$K \subset \bigcup_{j=1}^{N} G_j .$$

Then by the theorem of partition of unity,

$\exists\ \varphi_j \prec G_j$ such that

$$\sum_{j=1}^{N} \varphi_j = 1 \quad \text{on } K .$$

We obtain that

$$f = \sum_{j=1}^{N} f \cdot \varphi_j \quad \text{on } X .$$

Hence

$$\Lambda(f) = \sum_{j=1}^{N} \Lambda(f\varphi_j)$$

$$\leq \sum_{j=1}^{N} \mu_0(G_j) \quad (\text{since } f\varphi_j \prec G_j)$$

$$\leq \sum_{j=1}^{\infty} \mu_0(G_j) .$$